Day 1: The Probabilistic Method

Exercise 1. (Russia 1999)

In a certain school, every boy likes at least one girl. Prove that we can find a set S of at least half the students in the school such that each boy in S likes an odd number of girls in S.

Walkthrough:

(a) Flip a coin for every girl to determine whether she goes in S or not. What is the expected number of girls in S?

(b) Put every boy who likes an odd number of girls in S into S. What is the expected number of boys in S?

(c) What is the expected size of S? Conclude.

Exercise 2. Show that any graph G with m edges has a bipartite subgraph with $\geq \frac{m}{2}$ edges.

Walkthrough:

(a) Flip a coin on every vertex and define a corresponding bipartite subgraph.

(b) Show that the expected number of edges in the subgraph is $\frac{m}{2}$, and conclude.

Exercise 3. (USAMO 2012, problem 2)

A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

Walkthrough:

(a) Consider a random symmetry of the 432-gon formed by the points. How many are there? (Don't include the identity.)

(b) Color the red points that land on green points orange. What's the expected number of orange points? How many orange points are guaranteed to achievable?

(c) Modify and repeat step (b).

(d) Conclude.

Exercise 4. (Erdős)

Let $R(s)$ denote the Ramsey number of s, i.e., the smallest integer n for which, when one colors the edges of K_n either red or blue, there must be a monochromatic K_s.

Show that $R(s) > 2^{s/2}$ for $s \geq 3$.

Walkthrough:

(a) Let $n = \lfloor 2^{s/2} \rfloor$. Showing that $R(s) > n$ is showing that there existing a coloring of K_n with no monochromatic K_s. Randomly color each edge of K_n red or blue. What is the probability that a given set of s vertices forms a monochromatic K_s?

(b) Show that it suffices to show $\binom{n}{s} < 2^{\binom{s}{2}-1}$, and verify this is true by showing that $\binom{n}{s} < \frac{n^s}{2^s} < 2^{\binom{s}{2}-1}$.

Bonus Homework 5. Alice marks ten points in the plane. Is it always possible for Bob to place ten unit discs to cover all ten points, so that no two discs overlap?

Theorem 6 (Markov's Inequality). *If $X \geq 0$ is a random variable and $a > 0$,*

$$P(X \geq a) \leq \frac{1}{a} \mathrm{E}[X],$$

with equality iff $X \in \{0, a\}$.

Homework 7. Prove this.

Definition 8. The covariance of two variables X and Y is

$$\mathrm{Cov}(X, Y) := \mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])]$$

Homework 9. Show that $\mathrm{Cov}(X, Y)$ can also be written as $\mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]$.

Definition 10. The variance of a random variable X is

$$\mathrm{Var}(X) := \mathrm{Cov}(X, X) = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] = \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2.$$

Theorem 11 (Chebyshev's Inequality). *If X is a random variable and $a > 0$,*

$$P(|X - \mathrm{E}[X]| \geq a) \leq \frac{\mathrm{Var}(X)}{a^2},$$

with equality iff $X - \mathrm{E}[X] \in \{0, a, -a\}$.

Proof. This is the direct result of plugging in $(X - \mathrm{E}[X])^2$ for X and a^2 for a into Markov's Inequality (**6**). ☐

Exercise 12. (USAMO 2012, problem 6)

For integer $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$x_1 + x_2 + \ldots + x_n = 0, \qquad \text{and} \qquad x_1^2 + x_2^2 + \ldots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \ldots, n\}$, define

$$S_A = \sum_{i \in A} x_i.$$

(If A is the empty set, then $S_A = 0$.)

Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of $x_1, x_2, \ldots, x_n, \lambda$ does equality hold?

Walkthrough:

(a) Flip n coins and let $X_i = x_i$ if the i^{th} coin comes up heads, and $X_i = 0$ if it comes up tails. Let A be the set of indices of coins that came up heads. Write
$$\sum_{i=1}^{n} X_i = S_A.$$

(b) What is $\mathrm{E}[X_i]$? $\mathrm{E}[S_A]$? $\mathrm{Var}(X_i)$? $\mathrm{Var}(S_A)$?

(c) What does Chebyshev's inequality (**11**) say when you plug in 2λ?

(d) Show that $P(S_A \geq \lambda) = P(-S_A \geq \lambda)$.

(e) Conclude that the inequality given in the problem holds.

(f) What does the equality case of Chebyshev's inequality say about the equality case for this problem? Conclude.
