Extremal Graph Theory

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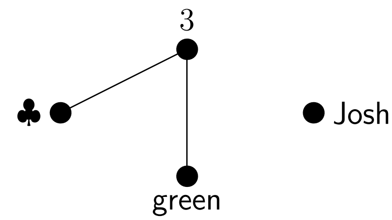
Definition 1. A *simple graph* is a pair (V, E) where V is a set and E is a subset of $\{\{u, v\} \mid u, v \in V, u \neq v\}$. We say that the elements of V are its *vertices* and the elements of E *edges*.
For a graph G we use $V(G)$ to denote the set of its vertices and $E(G)$ to denote the set of its edges.

Graphs are often drawn like networks, where vertices are represented as nodes and edges as lines connecting them.

The degree of a vertex v, denoted $d(v)$, is the number of edges containing v. In the diagram on the right, $d(3) = 2$ and $d(\text{Josh}) = 0$.



A labeled simple graph diagram of $(\{♣, \text{green}, \text{Josh}, 3\}, \{\{♣, 3\}, \{\text{green}, 3\}\})$.

Theorem 2. *The sum of all degrees in a graph is twice its number of edges, i.e.,*

$$\sum_{v \in V} d(v) = 2|E|.$$

Proof. The left side is the number of pairs (v, e) where $v \in V$ is a vertex and $e \in E$ is an edge containing v, as once one has chosen v, there are $d(v)$ ways to choose e.
The right side is the number of pairs (e, v) where $e \in E$ is an edge and $v \in V$ is a vertex in e, as once one has chosen e, there are two vertices in e between which to choose.
These are the same. □

A graph is isomorphic to another if they have identical unlabeled graph diagrams. (More formally, if there is a bijection between their vertex sets that matches their edges.) For our purposes, isomorphic graphs are considered identical.
Extremal combinatorics studies minimal and maximal objects satisfying certain properties. *Extremal graph theory* is the intersection between extremal combinatorics and *graph theory*, the study of graphs. We will look at three questions of the form "What is the maximum value of X in a graph satisfying Y?"

Ramsey Theory

Definition 3. The *empty graph* on n vertices is the graph with n vertices and no edges.

Definition 4. The *complete graph* on n vertices, K_n, is the graph with n vertices and an edge between every pair of vertices.

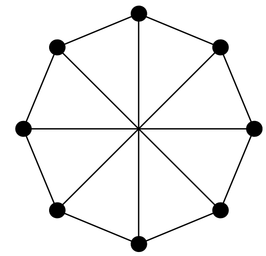
Definition 5. An *independent set* of a graph G is a subset S of its vertices that "forms" an empty graph, where there is no edge $\{u, v\}$ for any $\{u, v\} \in S$.

Definition 6. A *clique* of a graph G is a subset of its vertices S such that "forms" a complete graph, where there is an edge $\{u, v\}$ for all $\{u, v\} \in S$.

Theorem 7. *(Ramsey) For all $a, b \in \mathbb{N}$ if $|V(G)| = \binom{a+b-2}{a-1}$, G contains either an independent set of size a or a clique of size b (or both).*

Q1: What is the maximum number of vertices in a graph with no independent sets of size a nor cliques of size b?

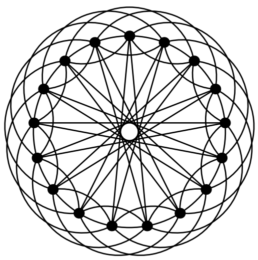
Definition 8. Maximal graphs meeting the requirements of Q1 are called *maximal (a, b)-Ramsey graphs.*



Exercise 9. Find the maximal (a, b)-Ramsey graph for $\min(a, b) = 2$.

Exercise 10. Find the maximal $(3, 3)$-Ramsey graph.

Exercise 11. Show that the graph on the right is a maximal $(4, 3)$-Ramsey graph.



Exercise 12. Show that the graph on the left is a maximal $(4, 4)$-Ramsey graph.

Hamiltonian Cycles

Definition 13. A *path* is sequence of vertices $v_1, \ldots, v_k$ of a graph G where v_i is connected to v_{i+1} for $1 \leq i < k$.

Definition 14. A *cycle* is a path $v_1, v_2, \ldots, v_k$ where $k > 2$ and v_k is connected back to v_1.

Definition 15. A *Hamiltonian path* of a graph G is a path that contains all vertices in $V(G)$.

Definition 16. A *Hamiltonian cycle* of a graph G a Hamiltonian path that is a cycle.

Definition 17. The *minimum degree* of a graph G, denoted $\delta(G)$, is the smallest degree of any of its vertices.

Q2: What is the maximum possible minimum degree $\delta(G)$ of a graph G with no Hamiltonian cycle?

Theorem 18. *(Dirac) If* $\delta(G) \geq \frac{|V(G)|}{2}$, *$G$ has a Hamiltonian cycle.*

Exercise 19. For all n, find a graph G on n vertices with no Hamiltonian cycle such that $\delta(G) \geq \frac{n}{2} - 1$.

Clique avoidance

Q3: What is the maximum possible number of edges $|E(G)|$ in a graph G with no cliques of size r?

Lemma 20. *If G does not have a clique of size r,* $\delta(G) \leq \frac{r-2}{r-1}n$.

Theorem 21. *(Turan) If G does not have a clique of size r,* $|E(G)| \leq \frac{r-2}{r-1} \cdot \frac{n^2}{2}$.